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Division algebras and extended $N = 2, 4, 8$ superKdVs

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Abstract
A no-go result for *integrable* minimal $N = 8$ supersymmetric extensions of KdV is found. However, allowing for non-associative realizations of the extended supersymmetries, the first example of an $N = 8$ supersymmetric KdV equation is explicitly constructed. It involves eight bosonic and eight fermionic fields and corresponds to the unique $N = 8$ solution based on a generalized Hamiltonian dynamics with (generalized) Poisson brackets given by the non-associative $N = 8$ superconformal algebra. The complete list of inequivalent classes of parametric-dependent $N = 3$ and $N = 4$ superKdVs obtained from the 'non-associative $N = 8$ SCA' is also furnished. Furthermore, a fundamental domain characterizing the class of inequivalent $N = 4$ superKdVs based on the 'minimal $N = 4$ SCA' is given.

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1. Introduction

The construction and classification of lower dimensional dynamical systems with extended (N) supersymmetries is a highly non-trivial problem, often investigated in connection with the dimensional reduction of supersymmetric field theories. In [1], developing some methods encountered in [2], a classification was presented for the matrix representations of 1D N-extended supersymmetry algebras. Besides the 1D N-extended supersymmetric quantum mechanical systems, this classification applies also to the extended supersymmetrizations of classical non-linear equations in 1 + 1 dimensions.

On the other hand, matrix representations of extended supersymmetries are not the end of the story. Indeed, an *inequivalent* realization of an $N = 8$ global supersymmetry makes use of the non-associative octonionic structure constants (see [3]). A classification of these non-associative realizations for generic N is now available [4].

The question we address in this paper is whether the non-associative realizations of supersymmetries can lead to as yet undiscovered supersymmetric extensions of non-linear equations in 1 + 1 dimensions (to be specific, we work with the simplest of such systems, the KdV equation).

It is worth mentioning that in the last several years bosonic integrable hierarchies of non-linear differential equations in 1 + 1 dimensions have been intensely explored, mainly in connection with the discretization of two-dimensional gravity (see [5]). Their supersymmetric extensions have also been largely investigated [6–12] using a variety of different methods. However, despite this activity, many questions are still unanswered.

The result of the present paper is negative, but still interesting and strong. It can be regarded as an unexpected improvement of a no-go theorem. It was already known, for reasons recalled later, that no superextension of KdV exists for $N > 4$ (the maximal superKdV is for $N = 4$). This result, however, was based on the implicit assumption that the extended supersymmetries were realized associatively. If we relax the condition of associativity and allow for non-associative realizations, we have quite a different picture. Here we are able to prove, this is the good news, that a non-associative $N = 8$ version of KdV indeed exists and, moreover, it is unique. The bad news is that this unique system, the 'non-associative $N = 8$ KdV', is not integrable.

Our strategy is based on checking how many supersymmetries can be implemented as invariance for the most general Hamiltonian of right dimension, admitting the 'non-associative $N = 8$ superconformal algebra' of Englert *et al* [13] as a generalized Poisson bracket structure. The non-associativity of this algebra (i.e. its failure in fulfilling the Jacobi identities) allows the presence of a central extension in its Virasoro subalgebra. For what concerns the ordinary N-extended superconformal algebras, the classification and the list of their central extensions have been produced in the mathematical literature [14]. Central charges are allowed for $N \leqslant 4$ only. The link with the (super)-KdV's equations is based on the fact that the third-order higher derivative term in the (super)-KdV equations is induced by the central extension of the Virasoro (sub)algebra. For that reason, only supersymmetric generalizations of KdV up to $N = 4$ (both integrable and non-integrable) have been constructed [7, 11] so far, consistently with the 'old' no-go theorem mentioned before.

The 'non-associative $N = 8$ SCA' involves eight bosonic and eight fermionic fields and is constructed in terms of octonionic structure constants. Its restriction to the real, complex or quaternionic subalgebras leads, respectively, to the ordinary $N = 1, 2, 4$ superconformal algebras (in the last case, this is the so-called 'minimal $N = 4$ SCA').

The paper is constructed as follows. First, the $N = 2, 4$ KdV equations are revisited in the language of division algebras, and a fundamental domain for the parametric space of the inequivalent $N = 4$ KdVs is produced (our results complete and complement the work of [11]).

Later, we apply the same techniques to investigate the global $N = 8$ invariance for the most general Hamiltonian of correct dimension constructed with the eight bosonic and eight fermionic fields entering the 'non-associative $N = 8$ SCA'. If we further assume invariance under the octonionic involutions, the Hamiltonian is unique up to the normalization factor, giving rise to a unique set of $N = 8$ KdV equations. The high price we have to pay for extending KdV up to $N = 8$ supersymmetries is its integrability. The non-integrability of the $N = 8$ KdV is manifest when reducing the sets of equations to their quaternionic subspace. As a consequence, the most symmetric (global $SU(2)$-invariant) set of $N = 4$ KdV equations is produced. This $N = 4$ KdV system, however, does not correspond to the unique point characterizing the integrable $N = 4$ KdV.

Following the authors of [15] who pointed out that global $N = 2$ supersymmetric systems can be obtained from the 'minimal $N = 4$ SCA' Poisson brackets, we extend here such an analysis by investigating the class of global $N = 3$ and $N = 4$ supersymmetric extensions of KdV which can be constructed with the fields satisfying the 'non-associative $N = 8$ SCA' generalized Poisson brackets. The complete solution is reported. In the $N = 4$ case two inequivalent classes (both parametric dependent) of solutions are found. The existence of two $N = 4$ classes is in consequence of the two inequivalent ways of associating three invariant supersymmetry charges with imaginary octonions (i.e. either producing, or not, an $su(2)$ subalgebra), while the extra supersymmetry charge is always associated with the octonionic identity. In the $N = 3$ case, just a single class of parametric solutions is found since any given pair of imaginary octonions is equivalent to any other pair.

Some comments are in order. Based on the Sugawara relation [16] concerning the 'non-associative $N = 8$ SCA' and the superaffined octonionic algebra, we can induce on the affine fields a global $N = 8$ set of equations, generalizing both the NLS and mKdV equations, as well as the $N = 4$ construction of [17].

We heavily relied on the Thieleman package for computing classical OPEs with Mathematica [18], supported by our own package to deal with octonionic structure constants.

2. On division algebras and the 'non-associative $N = 8$ SCA'

In this section we recall (see [16, 19]) the basic properties of the division algebra of the octonions which will be used in the following and introduce the 'non-associative $N = 8$ superconformal algebra' according to [13] (see also [16]).

A generic octonion x is expressed as $x = x_a\tau_a$ (throughout the text the convention over repeated indices, unless explicitly mentioned, is understood), where x_a are real numbers while τ_a denote the basic octonions, with $a = 0, 1, 2, \ldots, 7$.

$\tau_0 \equiv \mathbf{1}$ is the identity, while τ_α, for $\alpha = 1, 2, \ldots, 7$, denote the imaginary octonions. In the following a greek index is employed for imaginary octonions, and a latin index for the whole set of octonions (identity included).

The octonionic multiplication can be introduced through

$$\tau_\alpha \cdot \tau_\beta = -\delta_{\alpha\beta}\tau_0 + C_{\alpha\beta\gamma}\tau_\gamma \tag{1}$$

with $C_{\alpha\beta\gamma}$ a set of totally antisymmetric structure constants which, without loss of generality, can be taken to be

$$C_{123} = C_{147} = C_{165} = C_{246} = C_{257} = C_{354} = C_{367} = 1 \tag{2}$$

and vanishing otherwise.

It is also convenient to introduce, in the seven-dimensional imaginary octonions space, a 4-indices totally antisymmetric tensor $C_{\alpha\beta\gamma\delta}$, dual to $C_{\alpha\beta\gamma}$, through

$$C_{\alpha\beta\gamma\delta} = \tfrac{1}{6}\varepsilon_{\alpha\beta\gamma\delta\epsilon\zeta\eta}C_{\epsilon\zeta\eta} \tag{3}$$

(the totally antisymmetric tensor $\varepsilon_{\alpha\beta\gamma\delta\epsilon\zeta\eta}$ is normalized so that $\varepsilon_{1234567} = +1$).

The octonionic multiplication is not associative since for generic a, b, c we get $(\tau_a \cdot \tau_b) \cdot \tau_c \neq \tau_a \cdot (\tau_b \cdot \tau_c)$. However, the weaker condition of alternativity is satisfied. This means that, for $a = b$, the associator

$$[\tau_a, \tau_b, \tau_c] \equiv (\tau_a \cdot \tau_b) \cdot \tau_c - \tau_a \cdot (\tau_b \cdot \tau_c) \tag{4}$$

is vanishing.

The specialization of the octonionic indices to, let us say, $0, 1$ or $0, 1, 2, 3$ leads, respectively, to the complex number or to the division algebra of quaternions.

The octonionic algebra admits seven involutions, specified by the mappings

$$\tau_0 \mapsto \tau_0 \qquad \tau_p \mapsto \tau_p \qquad \tau_q \mapsto -\tau_q \tag{5}$$

where p takes a value in one of the seven triples entering (2), while q specifies the four complementary values. The three involutions for the quaternions (with two generators) are recovered as the restrictions to the 0, 1, 2, 3 subspace.

The $N = 8$ extension of the Virasoro algebra (non-associative $N = 8$ SCA) involves eight bosonic and eight fermionic fields and is constructed in terms of the octonionic structure constants. Besides the spin-2 Virasoro field denoted as T, it contains eight fermionic spin-$\frac{3}{2}$ fields Q, Q_α and seven spin-1 bosonic currents J_α. It is explicitly given by the following Poisson brackets:

$$
\begin{aligned}
\{T(x), T(y)\} &= -\tfrac{1}{2}\partial_y{}^3\delta(x-y) + 2T(y)\partial_y\delta(x-y) + T'(y)\delta(x-y)\\
\{T(x), Q(y)\} &= \tfrac{3}{2}Q(y)\partial_y\delta(x-y) + Q'(y)\delta(x-y)\\
\{T(x), Q_\alpha(y)\} &= \tfrac{3}{2}Q_\alpha(y)\partial_y\delta(x-y) + Q'_\alpha(y)\delta(x-y)\\
\{T(x), J_\alpha(y)\} &= J_\alpha(y)\partial_y\delta(x-y) + J'_\alpha(y)\delta(x-y)\\
\{Q(x), Q(y)\} &= -\tfrac{1}{2}\partial_y{}^2\delta(x-y) + \tfrac{1}{2}T(y)\delta(x-y)\\
\{Q(x), Q_\alpha(y)\} &= -J_\alpha(y)\partial_y\delta(x-y) - \tfrac{1}{2}J'_\alpha(y)\delta(x-y)\\
\{Q(x), J_\alpha(y)\} &= -\tfrac{1}{2}Q_\alpha(y)\delta(x-y)\\
\{Q_\alpha(x), Q_\beta(y)\} &= -\tfrac{1}{2}\delta_{\alpha\beta}\partial_y{}^2\delta(x-y) + C_{\alpha\beta\gamma}J_\gamma(y)\partial_y\delta(x-y)\\
&\quad + \tfrac{1}{2}(\delta_{\alpha\beta}T(y) + C_{\alpha\beta\gamma}J'_\gamma(y))\delta(x-y)\\
\{Q_\alpha(x), J_\beta(y)\} &= \tfrac{1}{2}(\delta_{\alpha\beta}Q(y) - C_{\alpha\beta\gamma}Q_\gamma(y))\delta(x-y)\\
\{J_\alpha(x), J_\beta(y)\} &= \tfrac{1}{2}\delta_{\alpha\beta}\partial_y\delta(x-y) - C_{\alpha\beta\gamma}J_\gamma(y)\delta(x-y).
\end{aligned}
\tag{6}
$$

Note the presence of the central term, essential in order to obtain supersymmetric KdV equations. Due to the non-associativity of octonions the structure constants of (6) do not satisfy the Jacobi identity (see [16] for a detailed discussion).

3. The $N = 2$ and the $N = 4$ KdVs revisited

By restricting the greek indices to take either the values 1 or 1, 2, 3, we recover from (6) the $N = 2$ and the $N = 4$ superconformal algebras, respectively (in the case of $N = 4$ the corresponding algebra is known as the 'minimal $N = 4$ SCA'). They can be regarded as one of the Poisson brackets for the $N = 2$ and the $N = 4$ KdVs [7, 11].

These non-linear equations can be constructed by looking for the most general Hamiltonian with the right dimension (i.e. whose Hamiltonian density has dimension 4) invariant under global supersymmetric charges given by $\int \mathrm{d}x\, Q(x)$ and $\int \mathrm{d}x\, Q_\alpha(x)$. This approach was used to construct the $N = 2$ KdV in [7], while the $N = 4$ KdV was obtained in terms of a harmonic superspace formalism in [11].

For what concerns the $N = 2$ case we summarize here the results of [7]. We avoid writing explicit formulae since they can be immediately recovered from a suitable reduction of the $N = 4$ KdV results as discussed later. Up to a normalization factor, the $N = 2$ invariant Hamiltonians depend on a single real parameter, denoted as 'a', which labels inequivalent $N = 2$ KdVs. Three special values for a, i.e. $a = -2, 1, 4$, correspond to the three inequivalent $N = 2$ KdV equations which are integrable [7, 10].

Here we extend the analysis of [7] to the $N = 4$ KdV case. In particular, we are able to fully determine the moduli space of inequivalent $N = 4$ KdVs. Our results extend and complete those originally appearing in [11].

The most general $N = 4$ invariant Hamiltonian of right dimension depends on five parameters (apart from the overall normalization factor) and is explicitly given by

$$
\begin{aligned}
H = \int \mathrm{d}x \Big[& -2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha + x_\alpha T J_\alpha^2 + 2x_\alpha Q Q_\alpha J_\alpha - \epsilon_{\alpha\beta\gamma} x_\gamma Q_\alpha Q_\beta J_\gamma \\
& + \frac{1}{3}\epsilon_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta J'_\gamma - 2z_\alpha \epsilon_{\alpha\beta\gamma} T J_\beta J_\gamma \\
& - 2z_1 Q(Q_2 J_3 + Q_3 J_2) - 2z_2 Q(Q_3 J_1 + Q_1 J_3) - 2z_3 Q(Q_1 J_2 + Q_2 J_1) \\
& + 2z_1 Q_1(Q_2 J_2 - Q_3 J_3) + 2z_3 Q_3(Q_1 J_1 - Q_2 J_2) + 2z_2 Q_2(Q_3 J_3 - Q_1 J_1) \\
& - z_1 J'_1\big(J_2^2 - J_3^2\big) - z_3 J'_3\big(J_1^2 - J_2^2\big) - z_2 J'_2\big(J_3^2 - J_1^2\big)\Big]
\end{aligned} \tag{7}
$$

where the convention over repeated indices is understood and α, β, γ are restricted to 1, 2, 3, while $\epsilon_{123} = 1$.

In order to guarantee the $N = 4$ invariance, the three parameters x_α must satisfy the condition

$$
x_1 + x_2 + x_3 = 0 \tag{8}
$$

so that only two of them are truly independent (together with the three z_α they provide the five parameters mentioned above). However, the further requirement for the Hamiltonian to be invariant not only under global $N = 4$ supersymmetry, but also under the three involutions of the $N = 4$ superconformal algebra (obtained by flipping the sign of the four fields J_α, Q_α, for $\alpha = 1, 2$, $\alpha = 1, 3$ and $\alpha = 2, 3$ respectively, while leaving unchanged the remaining four fields) kills the three z_α parameters, which must be set equal to zero.

The most general Hamiltonian of such a kind is, therefore, given by

$$
\begin{aligned}
H = \int \mathrm{d}x \Big[& -2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha + x_\alpha T J_\alpha^2 + 2x_\alpha Q Q_\alpha J_\alpha \\
& - \epsilon_{\alpha\beta\gamma} x_\gamma Q_\alpha Q_\beta J_\gamma + \frac{1}{3}\epsilon_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta J'_\gamma \Big]
\end{aligned} \tag{9}
$$

where of course (8) continues to hold.

Since any given ordered pair of the three parameters x_α can be chosen to be plotted along the x and y axes describing a real x–y plane, it can be easily proved that the fundamental domain of the moduli space of inequivalent $N = 4$ KdV equations can be chosen to be the region of the plane comprised between the real axis $y = 0$ and the $y = x$ line (boundaries included). Five other regions of the plane (all such regions are related via an S_3-group transformation) could as well be chosen as the fundamental domain.

In the region of our choice, the $y = x$ line corresponds to an extra global $U(1)$ invariance, since the Hamiltonian whose parameters live in this line is in involution with the global charge $\int \mathrm{d}x\, J_3$ (namely $\{H, \int \mathrm{d}x \cdot J_3\} = 0$). The origin, that is $x_1 = x_2 = x_3 = 0$, is the most symmetric point, corresponding to a global $SU(2)$ invariance, the given Hamiltonian being in involution with respect to the three $\int \mathrm{d}x \cdot J_\alpha$ charges.

The equations of motion for the whole class of inequivalent $N = 4$ KdVs are given by

$$
\begin{aligned}
\dot{T} = {} & -T''' - 12T'T - 6Q''Q - 6Q''_\alpha Q_\alpha + \left(4 + \frac{x_\alpha}{2}\right) J'''_\alpha J_\alpha + \frac{3}{2} x_\alpha J''_\alpha J'_\alpha + 3x_\alpha \big(T J_\alpha^2\big)' \\
& + 6x_\alpha (Q Q_\alpha J_\alpha)' - 3x_\gamma \epsilon_{\alpha\beta\gamma}(Q_\alpha Q_\beta J_\gamma)' + \epsilon_{\alpha\beta\gamma}(x_\gamma - x_\beta)(J''_\alpha J_\beta J_\gamma - J_\alpha J'_\beta J'_\gamma) \\
\dot{Q} = {} & -Q''' - 6(TQ)' - \left(4 + \frac{x_\alpha}{2}\right)(Q'_\alpha J_\alpha)' + \left(2 - \frac{x_\alpha}{2}\right)(Q_\alpha J'_\alpha)' + 3x_\alpha \big(Q J_\alpha^2\big)' \\
& - \epsilon_{\alpha\beta\gamma}(x_\gamma - x_\beta)(Q_\alpha J_\beta J_\gamma)'
\end{aligned}
$$

$$
\begin{aligned}
\dot{Q}_\alpha = {} & -Q_\alpha''' - 6(TQ_\alpha)' + \left(4+\frac{x_\alpha}{2}\right)(Q'J_\alpha)' - \left(2-\frac{x_\alpha}{2}\right)(QJ_\alpha')' + 3x_\beta\big(Q_\alpha J_\beta^2\big)' \\
& + \epsilon_{\alpha\beta\gamma}(x_\gamma - x_\beta)(QJ_\beta J_\gamma)' + \epsilon_{\alpha\beta\gamma}\left(4+\frac{x_\gamma}{2}\right)(Q_\beta' J_\gamma)' \\
& - \epsilon_{\alpha\beta\gamma}\left(2-\frac{x_\gamma}{2}\right)(Q_\beta J_\gamma')' + 2(x_\beta - x_\alpha)(1-\delta_{\alpha\beta})(J_\alpha Q_\beta J_\beta)' \\
\dot{J}_\alpha = {} & -J_\alpha''' - \left(4+\frac{x_\alpha}{2}\right)(TJ_\alpha)' + \left(2-\frac{x_\alpha}{2}\right)(QQ_\alpha)' - 2(x_\alpha + x_\beta)Q_\alpha Q_\beta J_\beta \\
& - \epsilon_{\alpha\beta\gamma}\left(1-\frac{x_\alpha}{4}\right)(Q_\beta Q_\gamma)' - 2\epsilon_{\alpha\beta\gamma}(x_\gamma - x_\beta)QQ_\beta J_\gamma \\
& + \epsilon_{\alpha\beta\gamma}\left(4+\frac{x_\gamma}{2}\right)(J_\beta' J_\gamma)' + 3x_\beta J_\alpha' J_\beta^2 + 2(1-\delta_{\alpha\beta})(x_\beta - x_\alpha)J_\alpha J_\beta' J_\beta
\end{aligned}
\tag{10}
$$

where the constraint $x_1 + x_2 + x_3 = 0$ is satisfied, and (x_1, x_2) takes a value either in the region $I \equiv \{x_1, x_2 | x_2 \geqslant x_1 \geqslant 0\}$ or in $II \equiv \{x_1, x_2 | x_2 \leqslant x_1 \leqslant 0\}$. Each given pair $(x_1, x_2) \in I \cup II$ labels an inequivalent $N = 4$ KdV equation.

The three involutions (each one associated with any given imaginary quaternion) allow us to perform three consistent reductions of the $N = 4$ KdV equation to an $N = 2$ KdV, by setting simultaneously equal to 0 all the fields associated with the τ which flip the sign (compare the discussion in the previous section). Therefore, the first involution allows us to consistently set equal to zero the fields $J_2 = J_3 = Q_2 = Q_3 = 0$, leaving the $N = 2$ KdV equation for the surviving fields T, Q, Q_1, J_1. Similarly, the second and the third involutions allow us to set equal to zero the four fields labelled by 1, 3 and 1, 2 respectively. It turns out that for each such reduction only one free parameter survives, namely, x_1, x_2 or respectively x_3.

This remaining free parameter coincides up to a normalization factor with the free parameter a of [7]. More specifically

$$
a = \tfrac{1}{4}x_\alpha \tag{11}
$$

with $\alpha = 1, 2, 3$ according to the reduction.

As a consequence, a necessary condition for the integrability of the $N = 4$ KdV requires that for a given pair $(x_1, x_2) \in I \cup II$ each one of the three reductions produces for a one of the known integrable values of a, namely, $-2, 1, 4$. It is then easily checked that there are only two points in $I \cup II$, both in the $U(1)$-invariant $x_1 = x_2$ line, implying integrability for the three reduced $N = 2$ KdVs. The solutions are

(i) $x_1 = x_2 = -8$, $(x_3 = 16)$ and
(ii) $x_1 = x_2 = 4$, $(x_3 = -8)$.

Only the first point, which produces the $a = -2$ and the $a = 4$ integrable $N = 2$ KdVs after reduction, corresponds to an integrable hierarchy [11].

4. The N = 8 superKdV

In this section we construct the first example of an $N = 8$ supersymmetric extension of the KdV equation. In order to be able to realize an $N = 8$ KdV we extend the method discussed in the previous section to the case of the 'non-associative $N = 8$ superconformal algebra' (6). The reason why we are forced to make use of a non-associative algebra has been discussed in the introduction.

More specifically, we started with the most general Hamiltonian of right dimension (its Hamiltonian density having dimension equal to 4) constructed with the 16 (8 bosonic and 8 fermionic) fields entering (6). Later we imposed some constraints on it. First we restricted the free coefficients in order to make the resulting Hamiltonian invariant under the whole set of seven involutions of the $N = 8$ superconformal algebra. This is the $N = 8$ extension of a requirement already encountered in the $N = 4$ case. The seven involutions are so defined. The fields T, Q are unchanged, as well as the six fields Q_α, J_α, for the α taking a value in one of the seven triples entering (2). The eight remaining fields Q_β, J_β, with β labelling the four complementary values (for any given choice of the original triple), have the sign flipped ($Q_\beta \mapsto -Q_\beta$, $J_\beta \mapsto -J_\beta$). After having constructed the most general Hamiltonian H invariant under the whole set of seven involutions, we started imposing the invariance under the $N = 8$ global supersymmetric transformations, that is we required

$$\left\{\int \mathrm{d}x \cdot Q_a(x), H\right\} = 0 \tag{12}$$

for $a = 0, 1, 2, \ldots, 7$ (here $Q_0 \equiv Q$), while $\{\star, \star\}$ denotes the generalized Poisson brackets given by the non-associative $N = 8$ SCA (6).

It is worth pointing out that for this generalized Hamiltonian system, the Poisson brackets are assumed to be classical. In particular, they satisfy the Leibniz property (or, better, its graded version due to the supersymmetry of (6)). The only feature of the non-associativity of the octonions lies in the non-vanishing of the Jacobi identities for the structure constants of the (6) algebra. The fields entering (6) are assumed to be ordinary (bosonic and fermionic) real fields.

Needless to say, to get the final answer we heavily relied on Mathematica's computations for classical OPEs, based both on the Thielemans package [18] and on our own package to deal with octonionic structure constants.

The final result is the following. There exists a unique Hamiltonian which is invariant under the whole set of global $N = 8$ supersymmetries. It admits no free parameter (apart from the trivial normalization factor) and is quadratic on the fields. It is explicitly given by

$$H_2 = -2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + 2J''_\alpha J_\alpha \tag{13}$$

(here $\alpha = 1, 2, \ldots, 7$ and the summation over repeated indices is understood). This result implies that there is only one $N = 4$ KdV system which can be consistently extended to $N = 8$ KdV, namely, the one which corresponds to the origin of the coordinates ($x_1 = x_2 = x_3 = 0$), that is the most symmetric point. While the corresponding Hamiltonian for the $N = 4$ case admits a global $SU(2)$ invariance, the $N = 8$ Hamiltonian (13) is invariant with respect to each one of the seven global charges $\int \mathrm{d}x \cdot J_\alpha(x)$, that is

$$\left\{\int \mathrm{d}x \cdot J_\alpha(x), H\right\} = 0. \tag{14}$$

The seven charges $\int \mathrm{d}x \cdot J_\alpha(x)$ generate a symmetry which extends $SU(2)$; it does not correspond to a group due to the non-associative character of the octonions.

Despite the apparent simplicity and the fact that it is quadratic in the fields, the Hamiltonian (13) generates an $N = 8$ supersymmetric extension of KdV which is not integrable, due to the fact that its $N = 4$ KdV reduction does not correspond to the integrable point of the $N = 4$ KdV.

The equations of motion of the $N = 8$ KdV are obtained through

$$\dot{\Phi}_i = \{\Phi_i, H\} \tag{15}$$

where Φ_i collectively denote the fields entering (6).

We explicitly obtain

$$\dot{T} = -T''' - 12T'T - 6Q''_a Q_a + 4J'''_\alpha J_\alpha$$
$$\dot{Q} = -Q''' - 6T'Q - 6TQ' - 4Q''_\alpha J_\alpha + 2Q_\alpha J''_\alpha - 2Q'_\alpha J'_\alpha$$

$$\dot{Q}_\alpha = -Q'''_\alpha - 2QJ''_\alpha - 6TQ'_\alpha - 6T'Q_\alpha + 2Q'J'_\alpha + 4Q''J_\alpha \\ - 2C_{\alpha\beta\gamma}(Q_\beta J''_\gamma - Q'_\beta J'_\gamma - 2Q''_\beta J_\gamma) \\ \dot{J}_\alpha = -J'''_\alpha - 4T'J_\alpha - 4TJ'_\alpha + 2QQ'_\alpha + 2Q'Q_\alpha - C_{\alpha\beta\gamma}(4J_\beta J''_\gamma + 2Q_\beta Q'_\gamma). \tag{16}$$

It is a simple exercise to prove that the equations of motion (16) are compatible with the $N = 8$ global supersymmetries generated by $\int \mathrm{d}x \cdot Q_a(x)$ $(a = 0, 1, 2, \ldots, 7)$ which provide the transformations

$$\delta_a \Phi_i(y) = \left\{ \int \mathrm{d}x \cdot Q_a(x), \Phi_i(y) \right\}. \tag{17}$$

The above system of equations corresponds to the first known example of an $N = 8$ supersymmetric extension of KdV.

5. On global $N = 3$ and $N = 4$ extended superKdVs based on the $N = 8$ SCA

The authors of [15] proved the existence of systems, obtained in terms of the $N = 4$ superconformal algebra, which admit only an $N = 2$ global supersymmetry.

It is worth considering in our context, which involves a larger number of supersymmetries, which kind of extended supersymmetric systems are supported by the non-associative $N = 8$ SCA. We present the complete analysis of the $N = 3$ and the $N = 4$ solutions. We construct the most general $N = 3$ and $N = 4$ superextensions of KdV admitting the non-associative $N = 8$ SCA as generalized Poisson brackets. Both such cases turn out to be parametric dependent.

Apart from the unique $N = 8$ solution, $N = 4$ is the largest number of supersymmetries which can be consistently imposed (by assuming an $N > 4$ invariance we automatically recover the full $N = 8$ invariance).

Both in the $N = 3$ and the $N = 4$ cases, without loss of generality, one of the invariant supersymmetric charges can always be assumed to be $\int \mathrm{d}x\, Q(x)$, with $Q(x)$ entering (6). In the $N = 3$ case the two remaining invariant supersymmetric charges (associated with imaginary octonions) can be chosen at will, since all pairs of imaginary octonions are equivalent. In the formula below, without loss of generality, we chose the invariant supersymmetric charges being given by $\int \mathrm{d}x\, Q_1(x)$ and $\int \mathrm{d}x\, Q_2(x)$.

The situation is different in the $N = 4$ case. Now we have three extra invariant supersymmetric charges to be associated with imaginary octonions. However, two inequivalent ways of choosing a triple of imaginary octonions exist, depending on whether the chosen triple corresponds to one of the seven values in (2) (i.e. the triples associated with an $su(2)$ subalgebra) or not. Two inequivalent classes of solutions, labelled by $N = 4$ (I) and $N = 4$ (II), are respectively obtained. The first (I) class can be individuated by choosing, without loss of generality, the three extra supersymmetric charges to be given by $\int \mathrm{d}x\, Q_1(x)$, $\int \mathrm{d}x\, Q_2(x)$ and $\int \mathrm{d}x\, Q_3(x)$. The second class (II), without loss of generality, can be produced by assuming invariance under $\int \mathrm{d}x\, Q_1(x)$, $\int \mathrm{d}x\, Q_2(x)$ and $\int \mathrm{d}x\, Q_4(x)$.

Let us present now the complete solutions.

The most general $N = 3$ invariant Hamiltonian depends (up to the normalization factor) on six free parameters entering x and x_τ $(\tau = 1, 2, \ldots, 7)$.

The seven x_τ satisfy two constraints

$$x_1 + x_2 + x_3 = 0 \qquad x_4 + x_5 + x_6 + x_7 = 0. \tag{18}$$

The most general Hamiltonian is given by

$$H = \int \mathrm{d}x \Big[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + xQ'_\mu Q_\mu + 2J''_\alpha J_\alpha - xJ''_\mu J_\mu + x_\alpha T J^2_\alpha x_\mu T J^2_\mu \\ + 2x_\alpha Q Q_\alpha J_\alpha + 2x_\mu Q Q_\mu J_\mu - x_\gamma C_{\alpha\beta\gamma} Q_\alpha Q_\beta J_\gamma - x_\nu C_{\alpha\mu\nu} Q_\alpha Q_\mu J_\nu \\ + (x_\mu + x_\nu) C_{\mu\nu\alpha} Q_\mu Q_\nu J_\alpha + \frac{1}{3} C_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta J'_\gamma + 2x_\mu C_{\alpha\mu\nu} J_\alpha J_\mu J'_\nu \Big] \tag{19}$$

where α, β, γ are restricted to take the values $1, 2, 3$, while μ, ν are restricted to the complementary values $4, 5, 6, 7$.

The equations of motion for this $N = 3$ generalization of KdV are directly computed from (19) by applying the Poisson brackets, as in (15).

The complete set of equations is written down in 37 pages of LaTex. For that reason they are not being reported here.

For what concerns the $N = 4$ cases, the (I) class of solutions involves three free parameters (up to the normalization factor) entering x and x_α ($\alpha = 1, 2, 3$), where the x_α are constrained to satisfy $x_1 + x_2 + x_3 = 0$.

The most general $N = 4$ invariant Hamiltonian of type (I) is given by

$$H = \int \mathrm{d}x \Big[-2T^2 - 2Q'Q - 2Q'_\alpha Q_\alpha + xQ'_\mu Q_\mu + 2J''_\alpha J_\alpha - xJ''_\mu J_\mu + x_\alpha T J^2_\alpha + x_\mu T J^2_\mu \\ + 2x_\alpha Q Q_\alpha J_\alpha + 2x_\mu Q Q_\mu J_\mu - x_\gamma C_{\alpha\beta\gamma} Q_\alpha Q_\beta J_\gamma - x_\nu C_{\alpha\mu\nu} Q_\alpha Q_\mu J_\nu \\ + (x_\mu + x_\nu) C_{\mu\nu\alpha} Q_\mu Q_\nu J_\alpha + \frac{1}{3} C_{\alpha\beta\gamma}(x_\beta - x_\alpha) J_\alpha J_\beta J'_\gamma + 2x_\mu C_{\alpha\mu\nu} J_\alpha J_\mu J'_\nu \Big]. \tag{20}$$

As before $\alpha, \beta, \gamma = 1, 2, 3$, while μ, ν take the values $4, 5, 6, 7$.

The $N = 4$ (I) equations of motion are explicitly given by

$$\dot{T} = -T''' - 12T'T - 6Q''Q - 6Q''_\alpha Q_\alpha + \left(4 + \frac{x_\alpha}{2}\right) J'''_\alpha J_\alpha + \frac{3}{2} x_\alpha J''_\alpha J'_\alpha + 3xQ''_\mu Q_\mu \\ - 2xJ'''_\mu J_\mu + 3x_\alpha \left(T J^2_\alpha\right)' + 6x_\alpha (Q Q_\alpha J_\alpha)' - 3x_\gamma C_{\alpha\beta\gamma}(Q_\alpha Q_\beta J_\gamma)' \\ + C_{\alpha\beta\gamma}(x_\gamma - x_\beta)(J''_\alpha J_\beta J_\gamma - J_\alpha J'_\beta J'_\gamma)$$

$$\dot{Q} = -Q''' - 6(TQ)' - \left(4 + \frac{x_\alpha}{2}\right)(Q'_\alpha J_\alpha)' + \left(2 - \frac{x_\alpha}{2}\right)(Q_\alpha J'_\alpha)' + 2x(Q'_\mu J_\mu)' \\ - x(Q_\mu J'_\mu)' + 3x_\alpha \left(Q J^2_\alpha\right)' - C_{\alpha\beta\gamma}(x_\gamma - x_\beta)(Q_\alpha J_\beta J_\gamma)'$$

$$\dot{Q}_\alpha = -Q'''_\alpha - 6(TQ_\alpha)' + \left(4 + \frac{x_\alpha}{2}\right)(Q' J_\alpha)' - \left(2 - \frac{x_\alpha}{2}\right)(Q J'_\alpha)' + 3x_\beta \left(Q_\alpha J^2_\beta\right)' \\ - 2xC_{\alpha\mu\nu}(Q'_\mu J_\nu)' + xC_{\alpha\mu\nu}(Q_\mu J'_\nu)' + C_{\alpha\beta\gamma}(x_\gamma - x_\beta)(Q J_\beta J_\gamma)' \\ + C_{\alpha\beta\gamma}\left(4 + \frac{x_\gamma}{2}\right)(Q'_\beta J_\gamma)' - C_{\alpha\beta\gamma}\left(2 - \frac{x_\gamma}{2}\right)(Q_\beta J'_\gamma)' \\ + 2(x_\beta - x_\alpha)(1 - \delta_{\alpha\beta})(J_\alpha Q_\beta J_\beta)'$$

$$\dot{Q}_\mu = \frac{x}{2} Q'''_\mu + (x - 4)TQ'_\mu - 6T'Q_\mu + 4Q''J_\mu + 2Q'J'_\mu + xQJ''_\mu + 4C_{\mu\alpha\nu}Q''_\alpha J_\nu \\ + 2C_{\mu\alpha\nu}Q'_\alpha J'_\nu + xC_{\mu\alpha\nu}Q_\alpha J''_\nu - 2xC_{\mu\nu\alpha}Q''_\nu J_\alpha - xC_{\mu\nu\alpha}Q'_\nu J'_\alpha \\ - 2C_{\mu\nu\alpha}Q_\nu J''_\alpha + x_\alpha C_{\mu\alpha\nu} Q Q_\alpha Q_\nu - x_\alpha C_{\mu\alpha\nu} Q J_\alpha J'_\nu \\ - 2x_\alpha C_{\mu\alpha\nu} Q J'_\alpha J_\nu - 2x_\alpha C_{\mu\alpha\nu} Q' J_\alpha J_\nu + 2x_\alpha Q'_\alpha J_\alpha J_\mu \\ + x_\alpha Q'_\mu J^2_\alpha + 3x_\alpha Q_\mu J'_\alpha J_\alpha - x_\alpha C_{\mu\nu\alpha} T Q_\nu J_\alpha + x_\alpha Q_\alpha J_\alpha J'_\mu + 2x_\alpha Q_\alpha J'_\alpha J_\mu$$

$$
\begin{aligned}
&+\frac{1}{2}C_{\mu\alpha\beta\nu}(x_\alpha+x_\beta)Q_\alpha Q_\beta Q_\nu-2x_\beta C_{\mu\alpha\beta\nu}Q'_\alpha J_\beta J_\nu+x_\alpha C_{\mu\nu\alpha\beta}Q_\nu J_\alpha J'_\beta\\
&-2x_\beta C_{\mu\alpha\beta\nu}Q_\alpha J'_\beta J_\nu-x_\beta C_{\mu\alpha\beta\nu}Q_\alpha J_\beta J'_\nu\\
\dot{J}_\alpha={}&-J'''_\alpha-\left(4+\frac{x_\alpha}{2}\right)(TJ_\alpha)'+\left(2-\frac{x_\alpha}{2}\right)(QQ_\alpha)'-2(x_\alpha+x_\beta)Q_\alpha Q_\beta J_\beta\\
&-C_{\alpha\beta\gamma}\left(1-\frac{x_\alpha}{4}\right)(Q_\beta Q_\gamma)'+xC_{\alpha\mu\nu}Q'_\mu Q_\nu-2C_{\alpha\beta\gamma}(x_\gamma-x_\beta)QQ_\beta J_\gamma\\
&+C_{\alpha\beta\gamma}-\left(4+\frac{x_\gamma}{2}\right)(J'_\beta J_\gamma)'-2xC_{\alpha\mu\nu}J''_\mu J_\nu+3x_\beta J'_\alpha J^2_\beta\\
&+2(1-\delta_{\alpha\beta})(x_\beta-x_\alpha)J_\alpha J'_\beta J_\beta\\
\dot{J}_\mu={}&\tfrac{1}{2}xJ'''_\mu-4(TJ_\mu)'+2Q'Q_\mu-xQQ'_\mu-2C_{\mu\alpha\nu}Q'_\alpha Q_\nu+xC_{\mu\alpha\nu}Q_\alpha Q'_\nu-4C_{\mu\nu\alpha}J''_\nu J_\alpha\\
&+2xC_{\mu\alpha\nu}J_\alpha J''_\nu+2x_\alpha C_{\mu\alpha\nu}TJ_\alpha J_\nu-x_\alpha C_{\mu\nu\alpha}QQ_\nu J_\alpha+2x_\alpha C_{\mu\alpha\nu}QQ_\alpha J_\nu\\
&+x_\alpha Q_\alpha J_\alpha Q_\mu+x_\alpha J'_\mu J^2_\alpha+2xJ'_\alpha J_\alpha J_\mu+2x_\alpha C_{\mu\alpha\beta\nu}J_\alpha J'_\beta J_\nu\\
&+x_\beta C_{\mu\alpha\beta\nu}Q_\alpha J_\beta Q_\nu+(x_\alpha+x_\beta)C_{\mu\alpha\beta\nu}Q_\alpha Q_\beta J_\nu.
\end{aligned}
\tag{21}
$$

The second (II) class of $N = 4$ solutions is two parametric. The free parameters can be chosen to be x_1 and x_2, while the remaining x_τ parameters entering the Hamiltonian below are restricted to be

$$x_3 = x_4 = -(x_1 + x_2) \qquad x_5 = 0 \qquad x_6 = x_1 \qquad x_7 = x_2. \tag{22}$$

The most general $N = 4$ (II) Hamiltonian is given by

$$
\begin{aligned}
H=\int \mathrm{d}x\,\Big[&-2T^2-2Q'Q-2Q'_\alpha Q_\alpha+2J''_\alpha J_\alpha+x_\alpha TJ^2_\alpha+2x_\alpha QQ_\alpha J_\alpha\\
&+C_{\rho\sigma\lambda}(x_\rho+x_\sigma)Q_\rho Q_\sigma J_\lambda+C_{\rho\lambda\sigma}(x_\rho+x_\lambda)Q_\rho Q_\lambda J_\sigma-C_{\lambda\mu\nu}(x_\lambda+x_\mu)Q_\lambda Q_\mu J_\nu\\
&+C_{\lambda\mu\rho}(x_\lambda+x_\mu)Q_\lambda Q_\mu J_\rho+2x_\mu C_{\lambda\mu\rho}Q_\lambda J_\mu Q_\rho-2x_\rho C_{\rho\lambda\sigma}J_\rho J_\lambda J'_\sigma\\
&+\frac{1}{3}C_{\mu\nu\lambda}(x_\mu-x_\nu)J_\mu J_\nu J'_\lambda-2x_\mu C_{\mu\rho\nu}J_\mu J_\rho J'_\nu\Big]
\end{aligned}
\tag{23}
$$

where now $\alpha = 1, 2, \ldots, 7$, while $\rho, \sigma = 1, 2, 4$ and $\lambda, \mu, \nu = 3, 5, 6, 7$.

The complete set of equations of motion for the $N = 4$ (II) case occupies 13 pages of LaTex. Just like the $N = 3$ case and contrary to the $N = 4$ (I) case, these equations of motion cannot be easily compactified since the field labels $1 \leftrightarrow 2, 3, 4, 5$ and $6 \leftrightarrow 7$ all play a different role.

Let us conclude this section with a final comment. The two classes (I) and (II) of $N = 4$ solutions are obviously inequivalent. For what concerns the first class we can notice that by suitably choosing the parameters x_α being given by $x_1 = x_2 = -8(x_3 = 16)$, the resulting generalized KdV system extends the integrable $N = 4$ KdV based on the 'minimal $N = 4$ SCA'. This leaves the possibility that the $N = 4$ (I) KdV, for the given values of the x_α parameters and for some $x \neq 0$, could be an integrable system.

6. Conclusions

In this paper, we addressed the problem of whether large N supersymmetric extensions of the KdV equation could be obtained from the non-associative, octonionic realizations of the extended supersymmetries [3, 4]. The 'non-associative $N = 8$ superconformal algebra' introduced in [13] (the 'non-associativity' referring to the fact that this superalgebra does not satisfy the super-Jacobi identities) was investigated as a possible generalized super-Poisson brackets structure for an $N = 8$ KdV, with eight bosonic and eight fermionic fields. Our results can be stated as follows. A previous no-go theorem, discarding the possibility of extended superKdVs for $N > 4$, on the basis of the absence of the central extensions for the (associative) superconformal algebras with $N > 4$ [14], is overcome. A new no-go theorem takes its place. A non-associative $N = 8$ KdV system indeed exists and is unique. However, it is not integrable. We further investigated and classified the N supersymmetric extensions (for $N > 2$) of KdV supported by the 'non-associative $N = 8$ SCA' generalized Poisson brackets. The complete results are reported here. Besides the unique $N = 8$ case, extensions are found for $N = 3$ and $N = 4$.

The class of solutions of the $N = 3$ case depends on six free parameters and is given in formula (19). For what concerns the $N = 4$ cases, two inequivalent classes of solutions, named '(I)' and '(II)', are found. The first class depends on three free parameters, while the second one depends on just two free parameters. They are given in formulae (20) and (23), respectively. For a convenient choice of the parameters of the class (I) solution, the resulting system of equations generalizes the integrable point of the 'minimal' $N = 4$ KdV, leaving room for the possibility that a global $N = 4$ system involving the whole set of $N = 8$ SCA fields could correspond to an integrable hierarchy.

Concerning the issue of integrability, some further remarks are needed. The non-integrable $N = 8$ KdV is unique and cannot be deformed (its Hamiltonian has been determined from the most general class of Hamiltonians with the right dimension, by brute force computer-aided methods). It corresponds to a minimal extension of the $N = 8$ KdV (i.e. with minimal number, eight, of bosonic and fermionic fields). At this stage of the investigation the possibility of non-minimal integrable $N = 8$ KdVs with a larger number of bosonic and fermionic fields cannot be ruled out. In this case, however, it is not even clear whether a Poisson brackets structure with non-trivial central charges generalizing the 'non-associative $N = 8$ SCA' even exists. Another open possibility concerns the non-linear realizations of the extended supersymmetries. In this case as well very little can be said since one cannot rely on the mathematical classifications available for the 'old' (associative) and 'new' (non-associative) no-go theorems.

Finally, it is worth mentioning that at least in one work [20] an octonionic model is discussed and proven integrable. It deserves investigation whether the techniques developed there could help in addressing the issue of integrability for the global $N = 4$ system given in formula (20).

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